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Lovata, Albert F.

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by

Albert F. Lovata

Lieutenant, United States Navy

Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE
IN
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SUBJECTED TO INTERNAL PRESSURE

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This work is accepted as fulfilling

the thesis requirements for the degree of

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IN

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ABSTRACT

The buckling action of hollow cylinders, subjected to internal pressure, was investigated under various conditions of axial load in the cylinder walls. The theory of Euler was found to apply where the modulus of elasticity and second moment of cross-sectional area are determined solely by the material and cross-section of the cylinder, and the load is the total axial load on the column.

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SYMBOLS

A_c Cross-sectional area of the cylinder (in^2).

A_p Cross-sectional area of the piston ($in.^2$).

a Lateral deflection of the mid-point of the column (.001 in.).

E Modulus of elasticity of the material of the cylinder (psi).

I Second moment of cross-sectional area about a centroidal axis ($in.^4$).

ℓ Length of the column (in.).

M Bending moment (in.-lbs.).

P Axial Compressive load (lbs.).

P_{cr} Critical load (lbs.).

p_i Internal pressure (psi.).

$(p_i)_{cr}$ Critical internal pressure (psi.).

r_c Radius of curvature (in.).

ρ Radius of gyration (in.).

I Introduction

1. Introduction.

A long cylindrical shape subjected to a compressive load will fail by sudden bending, provided the member is slender. This is the type of loading investigated by Euler [1] who determined a "critical load" such that any load greater than this will cause the column to bend. In his analysis Euler considered the column to be solid and subjected to an axial compressive load applied at the ends.

Figure 1. Schematic Representation of Column

Assume a hollow cylinder filled with a fluid capable of being placed under pressure (Fig. 1). One end of the cylinder is rigidly sealed and the other is sealed by means of a frictionless piston. Let the fluid be placed under pressure by means of an axial compressive load P applied at the ends. Symbolically the theory of Euler states: if P_{cr} is the critical load; E the modulus of elasticity of the material; I the second moment of the cross-sectional area about a centroidal axis; and ℓ the

the length of the column in the hinged-end configuration; then

$$P_{cr} = \frac{\pi^2 E I}{l^2}$$

A free body diagram of the column of Fig. 1 is identical to the free body diagram of any column with the same external restraints. Applying Euler's equation to the column of Fig. 1 results in: The load being the product of the piston area A_p and the internal pressure p_i. The modulus of elasticity and the second moment of the cross-sectional area are determined solely by the material and the cross-section of the cylinder. Writing Euler's equation in terms of a hollow cylinder filled with fluid results in

$$(p_i)_{cr} A_p = \frac{\pi^2 E I}{l^2}$$

In the undeflected condition the entire axial load of the column of Fig. 1 is carried by the fluid. Thus we are led to the interesting paradox of a column, which in the undeflected condition sustains no axial load, buckling when the internal pressure reaches a certain critical value.

Interest was focused on this problem when a long cylinder used in the hydraulic system of a submarine was observed to buckle when the piston approached the end of its stroke. With the increase in operating pressure of hydraulic plants and the associated decrease in size of component parts, it is conceived that in some applications the buckling action of cylinders subjected to internal pressures will be important.

2. Object and Scope

The object of this investigation was to verify the applicability of the theory of Euler to hollow cylinders subjected to internal pressure. It was previously shown that

$$(p_\lambda)_{cr} = \frac{\pi^2 E I}{A_p \ell^2}$$

This equation states that, for a given length, the internal pressure which will cause columns of the same material to buckle is proportional to the quotient of the second moment of cross-sectional area by the area of the piston.

The axial load carried by the cylinder is determined by the relation between the area of the piston and the area of the inside of the cylinder. The area of the inside of the cylinder does not appear in the equation. Therefore it may be stated that, theoretically, the critical pressure is independent of the "load-ratio." (Load-ratio is defined as the ratio of the axial load carried by the cylinder to the total axial load on the column.)

To verify this conclusion it was planned to test three sets of columns. In the first set the inner area of the cylinder was less than that of the piston (a positive load-ratio). In the second set the areas were equal (a zero load-ratio). In the third set the inner area of the cylinder was greater than that of the piston (a negative load-ratio).

To check the validity of the theory through a range of pressures, the lengths of the column were changed so that slenderness-ratios from 100 to 200 were obtained. The effect of these variables was studied experimentally and the results correlated with theory.

3. Acknowledgments.

The experimental work was carried out at the United States Naval Postgraduate School, Monterey, California, during the 1957-1958 academic year. The author is indebted to Professor R. E. Newton for his guidance

and assistance during all phases of the project and to Mr. R. P. Kennicott for his assistance in design and manufacture of the equipment used.

II Theory

Consider a straight cylindrical tube of homogeneous material, which obeys Hooke's Law, filled with a fluid capable of being placed under pressure. One end of the tube is rigidly sealed and the other end is sealed by means of a frictionless piston. The two ends are subjected to an externally applied axial load. The ends are restrained against lateral movement but have rotational freedom. The free body diagram of the column (Fig. 1) is identical in form to the one considered in Euler's classic work.

From Euler's solution it is known that under an axial compressive loading of magnitude P, applied at the centroids of the end cross-sections, the tube will remain straight until P reaches a certain critical value. As P exceeds this critical value the tube will begin to bend laterally in a plane containing the axis of the tube. If the tube has a circular cross-section this plane will not be known in advance. The lateral bending increases rapidly with P, so that it is practical to consider that the critical value of P represents the maximum load that the column can support. In order to evaluate this critical load we choose a set of cartesian coordinates, with the x-axis the locus of centroids of the cross-sections of the undeformed tube, the xy-plane containing the axis of the deflected tube, and the z-axis perpendicular to the x and y axes. The origin is at one end of the tube. If we consider the column in a deflected position (Fig. 2) it is clear that the magnitude of the bending moment at a particular cross-section is Py where the y-coordinate is that of the centroid of the section.

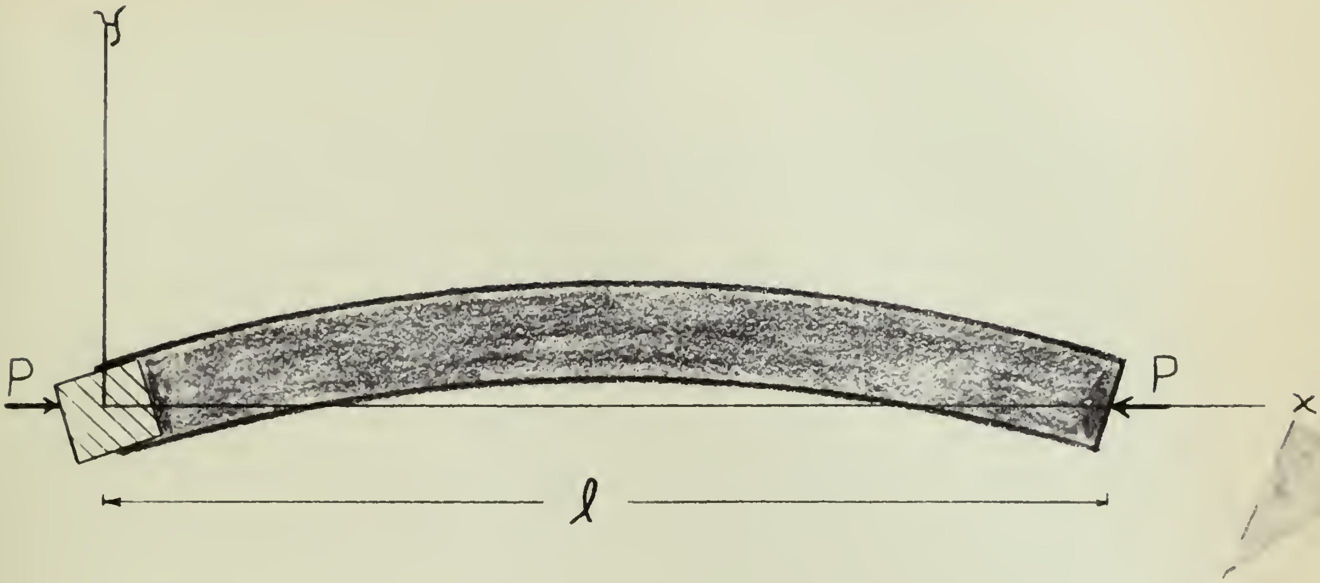


Figure 2. Free-body Diagram of Column in Deflected Condition

It is known from the elementary theory of flexure that the curvature produced by a bending moment is equal to the quotient of the moment by the flexural rigidity of the tube. The flexural rigidity is E I. From which

$$\frac{1}{r_c} = \frac{M}{E\,I},$$

where r_c is the radius of curvature

Now $$\frac{1}{r_c} = \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}$$

The condition of particular interest is the one in which the column first starts to deflect. At this time the slope is small and neglecting it will produce a negligible error and greatly simplify the solution.

Thus we have

$$\frac{1}{r_c} \cong \frac{d^2y}{dx^2} .$$

Employing a consistent set of algebraic signs we are led to the differential equation

$$\frac{d^2y}{dx^2} + \frac{P}{EI} y = 0 .$$

The general solution of this equation is

$$y = A \cos kx + B \sin kx, \quad \text{where} \quad k^2 = \frac{P}{EI} .$$

The end conditions are satisfied if

$$A = 0$$

and either: $B = 0$

or $\sin kl = 0$

If $B = 0$ then $y = 0$ and this is the solution prior to buckling.

If $\sin kl = 0$ then

$$kl = n\pi,$$

or $l\left(\frac{P}{EI}\right)^{1/2} = n\pi,$

or $P_{cr} = \frac{n^2\pi^2 EI}{l^2}$ where n is an integer.

The solution of principal interest is the one obtained for $n = 1$, from which

$$P_{cr} = \frac{\pi^2 EI}{l^2}$$

(The solution obtained for $n = 0$ is that of the undeformed column. Solutions obtained for values of n greater than one require that the

column pass through a region of instability and are thus possible only in theory or with the use of external constraints.)

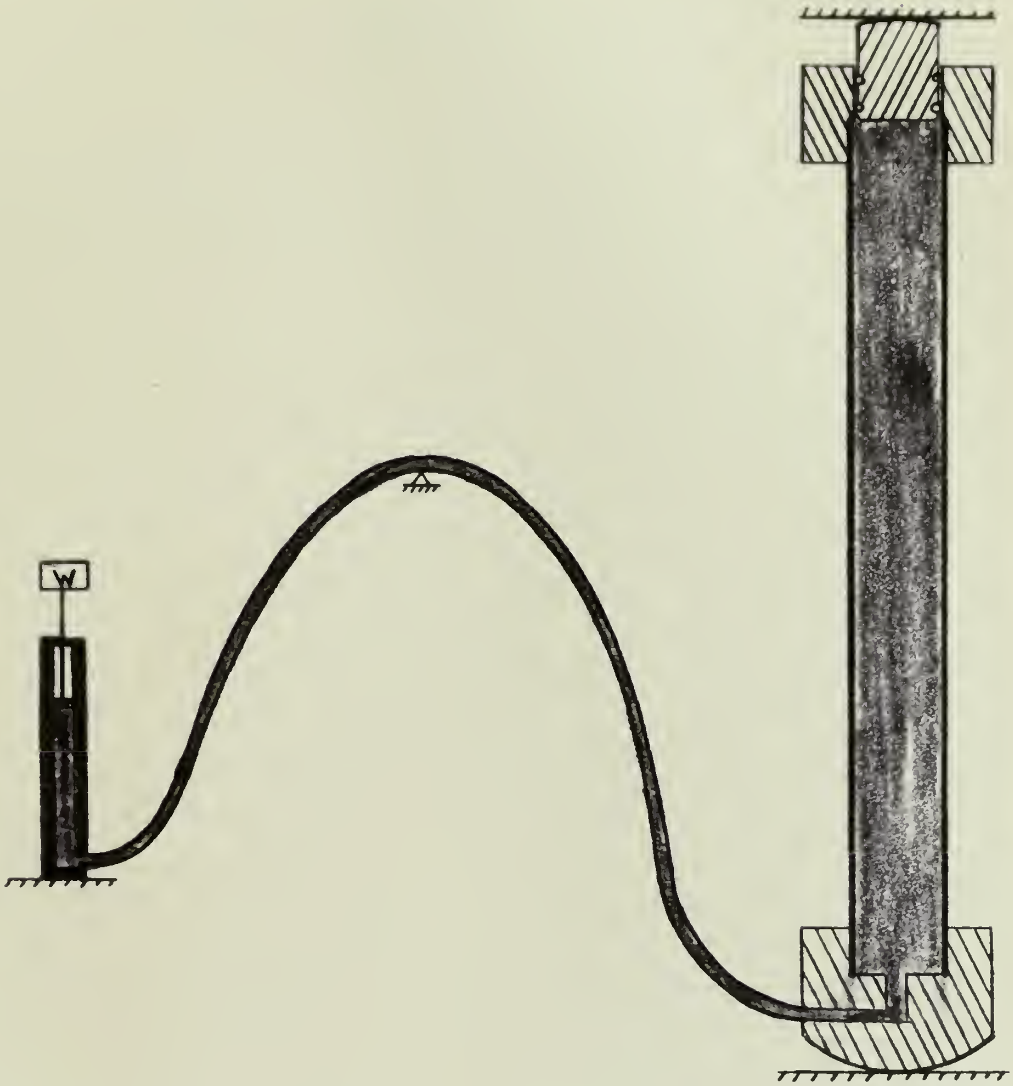


Figure 3. Schematic Representation of the Experimental Set-up

Figure 3 is a schematic representation of the experimental set-up. If the piston is frictionless then P may be replaced by its equivalent in terms of internal pressure $p_i A_p$. The fluid can not sustain shear so that the modulus of elasticity and the second moment of cross-sectional

area depend only on the material and cross-section of the cylinder. Placing the equation in terms of the internal pressure we obtain

$$(p_i)_{cr} = \frac{\pi^2 E A_c}{\left(\frac{l}{\rho}\right)^2 A_p} .$$

Where A_c is the area of the cross-section, and ρ is the radius of gyration of the cross-section of the cylinder.

III Description of Apparatus

1. Selection of Material

The buckling load, and thus the internal pressure producing buckling, is directly proportional to the modulus of elasticity. Aluminum alloy has the lowest modulus of elasticity among the readily available tubing materials and was therefore chosen.

2. Choice of Geometrical Properties

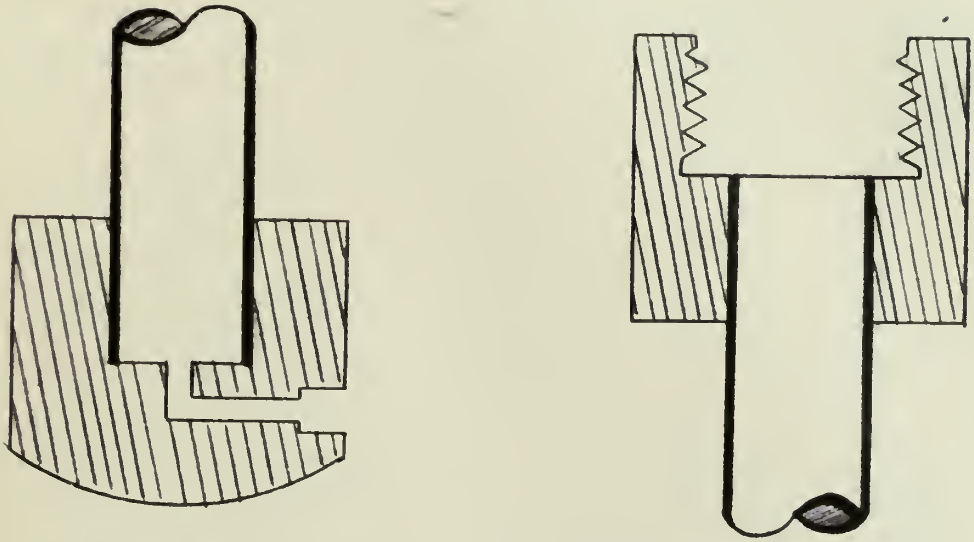
The principal limitation placed on the geometric properties was that the tube buckle before the hoop tension reached the yield point. A slenderness-ratio of 100, which fulfills this requirement, was selected as the minimum to be tested.

Establishing a minimum slenderness-ratio and a maximum internal pressure (see Section 5) dictates a maximum diameter-to-thickness ratio. Among the geometrically similar tubes which could be used outer diameters of 3/4, 1 and 1-1/4 inch were selected because they provide lengths which could be conveniently handled. A thickness of .035 inch was required for the 1-1/4 inch tube. This thickness also provided satisfactory characteristics for the other two sizes.

3. End Fittings

Fig. 4 is a schematic diagram of the end fittings. The tube and end fittings are joined by extra fine threads, 56 threads per inch. The upper end fitting has a cylinder joined to it by threads. (Henceforth this will be considered one piece and called "the upper fitting.") The upper portion of the piston, which contacts the horizontal surface of the testing machine, is a spherical surface (radius 2.75 inches). The center of this surface is on the axis of the piston in the plane of the face of the upper fitting. (Face of a fitting is defined as that flat

surface through which the tube is threaded.)



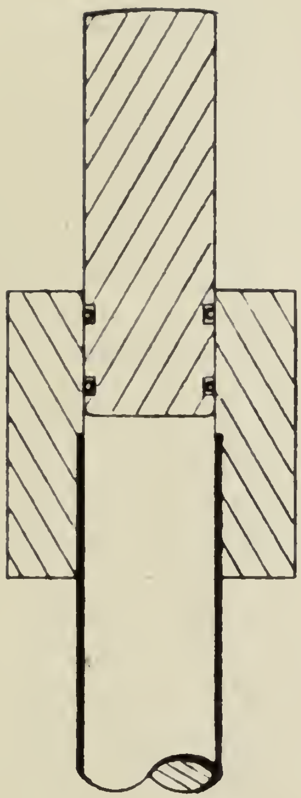
a) Lower b) Upper

Figure 4. End Fittings

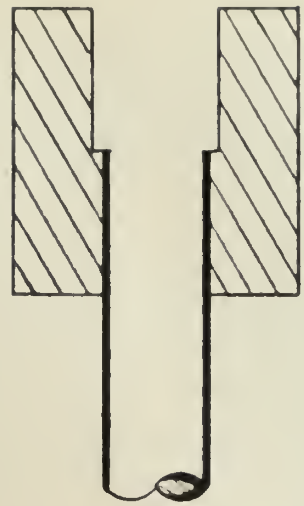
The lower fitting is similarly formed with the center of the spherical surface (radius 1.75 inches) on its axis in the plane of its face. The lower fitting has a passage, fitted with a 1/4 inch pipe nipple, to allow entry of the hydraulic fluid.

The effect of the spherical surfaces is to cause the column to act as if it were in the hinged-end configuration with an effective length equal to the distance between the faces.

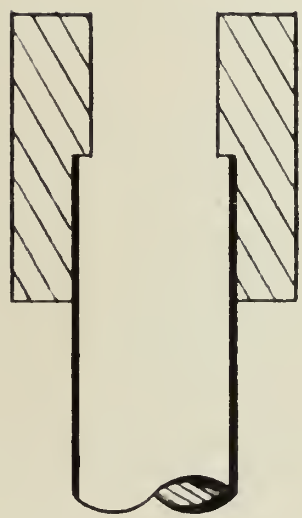
The piston and upper fittings (Fig. 5-a) retain the hydraulic fluid and dictate the axial load transmitted to the tube. The piston is fitted with an O-ring seal to limit leakage. The same piston is used in testing the three different sizes of columns. This results in a positive load-ratio, a zero load-ratio or a negative load-ratio;



a) Piston, and upper fitting with 1 inch tube



b) Upper fitting with 3/4 inch tube



c) Upper fitting with 1-1/4 inch tube

Figure 5. Cylinder, Piston, and End Fitting

depending on whether the inner diameter of the tube is less than (Fig. 5-b), equal to (Fig. 5-a), or greater than (Fig. 5-c) the diameter of the piston. (The diameter of the piston is taken to be the inner diameter of the cylinder.)

4. Deflection Measurement

Lateral deflections were measured by means of two dial gages set at right angles to each other and mounted, at the mid-point of the column, in a plane normal to the axis of the undeformed column.

5. Pressure Application

The pressure was applied by a dead-weight gage testing machine, using hydraulic fluid. The apparatus was designed to carry pressures to 2000 psi. It was felt that pressures greater than this would complicate the sealing of the threaded section and require a more complex O-ring design.

The fluid is carried from the gage tester to the lower fitting through 10 feet of hose having an inner diameter of 3/16 inch and an outer diameter of 33/64 inch. The hose consists of a synthetic rubber inner tube, a single wire braid reinforcement, and a synthetic rubber cover.

Fig. 6 shows a photograph of the apparatus assembled and ready for test.

Figure 6. Photograph of the Apparatus

IV Experimental Procedure

In deriving the theory it was assumed that the piston was frictionless. The experimental procedure was devised to eliminate the effects of friction between the piston and cylinder. The first method attempted was to apply the pressure, then turn the piston within the cylinder. This method appeared to eliminate the friction, but the resulting lateral movement of the column made the lateral deflection readings unreliable. The method finally used to eliminate the frictional effect was to apply the pressure and then adjust the testing machine to a load equal to the product of the piston area by the applied pressure. The column was aligned in a vertical position by using the plumb bubbles on a 28-inch carpenter's level.

The sequence of operation was as follows:

1) The column was placed in a vertical position by means of the level;
2) a fraction of the critical pressure was applied by loading the gage tester;
3) the load on the testing machine was adjusted to that load which it would carry if the piston were frictionless;
4) the lateral deflections were read from the dial gages;
5) the piston displacement was checked to insure that it had not reached the end of its travel;
6) the pressure applied by the gage tester was increased.

Steps 3) through 6) were repeated until the deflection read in Step 4) reached the limit of the dial gage. The critical pressure was

obtained using a method suggested by Southwell (Appendix I, Section 1).

V Results and Conclusions

1. Tabular Results

The theoretical and experimental values of critical pressure and unit load (unit load is the quotient of the total axial load on the column by the cross-sectional area of the cylinder) are summarized in Table 1. Additional data appears in Appendix I, Section 4.

2. Graphical Results

The theoretical curve of unit load versus slenderness-ratio, with experimental points plotted, is given in Fig. 7.

3. Conclusions

The experimental data indicate that the equation of Euler

$$P_{cr} = \frac{\pi^2 E I}{\ell^2}$$

is applicable to hollow cylindrical shapes subjected to internal pressure, where: P is the total axial load on the column; and the modulus of elasticity and second moment of cross-sectional area depend solely on the material and cross-section of the cylinder.

Table 1

Column Number	Slenderness ratio	Critical Pressure (psi) Theoretical	Critical Pressure (psi) Experimental	Unit Load Theoretical	Unit Load Experimental	% Difference
		3/4 inch tube				
1	105.1	1092	1074	9460	9300	-1.7
2	105.1	1093	1091	9460	9450	-0.2
3	120.2	835	820	7220	7090	-1.9
4	120.2	835	819	7220	7080	-1.9
5	147.8	555	543	4790	4690	-2.2
6	147.8	553	543	4790	4700	-1.8
7	202.6	296	296	2550	2550	0
8	202.6	297	292	2550	2510	-1.7
		1 inch tube				
9	100.5	1643	1625	10360	10250	-1.1
10	100.5	1646	1649	10360	10380	+0.2
11	118.7	1172	1189	7430	7540	+1.4
12	118.1	1190	1194	7500	7520	+0.3
13	142.2	816	800	5180	5070	-2.0
14	142.3	816	811	5170	5140	-0.6
15	185.6	474	465	3040	2980	-1.9
16	185.5	475	474	3040	3030	-0.5
		1-1/4 inch tube				
17	101.5	2068	2012	10160	9890	-2.7
18	101.5	2069	2024	10160	9940	-2.2
20	118.0	1508	1503	7510	7480	-0.3
21	136.3	1138	1120	5630	5540	-1.6
22	136.3	1136	1118	5630	5550	-1.6
23	152.4	917	917	4510	4510	0
24	152.4	919	917	4510	4510	-0.2

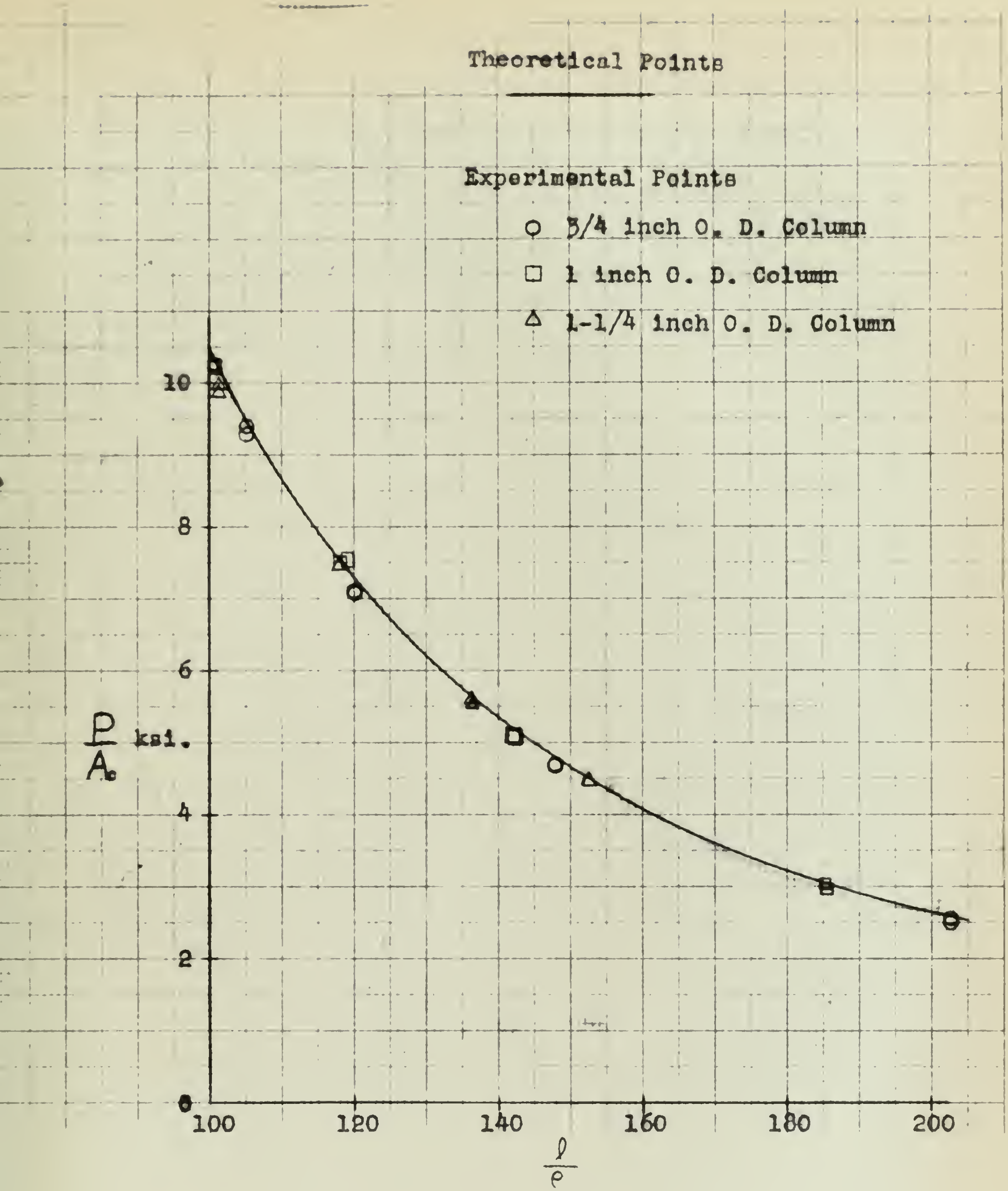


Figure [illegible]. Theoretical Curve with Experimental Points

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APPENDIX I

ANALYSIS OF EXPERIMENTAL DATA

1. Southwell's Method

In the ideal case of column buckling there would be no lateral deflection up to the critical value of the load. At the critical load the column would then begin to deflect. Owing to various kinds of imperfections, such as some unavoidable initial curvature of the column, eccentricity in application of the load and non-homogeneity of the material, the column begins to deflect with the beginning of loading and usually fails before Euler's load is reached.

A useful method of determining the critical load from the test data was proposed by R. V. Southwell [2]. Assume that the deflection of a column, under a load that is below the critical value, is due to initial curvature and eccentricity of load application. If the deflection is expressed as a trigonometric series, when the load approaches the critical value the first term in the series predominates.

If the deflection d is represented by the first term of the series, the following linear relation exists

$$d = P_{cr} \frac{d}{P} - a$$

where a is a constant. A cartesian plot of d versus $\frac{d}{P}$ yields a curve whose slope asymptotically approaches the critical load.

The experimental data of Column 11 are plotted in Figs. 8 through 12. The nature of the curve and its asymptotic behavior can be observed.

2. Sample Calculations

The critical load used in the theoretical derivation is the product of the internal pressure and the piston area. In the experimental procedure it was advantageous to consider the internal pressure to be the principal variable. For this reason the data are reduced in terms of the internal pressure.

Column 11   Run 1

	Gage 1		Gage 2	
p	d	d/p	d	d/p
200	2.5		5	.025
400	3.5		13	.032
600	4.5		22	.037
700	5		32	.046
800	6		42	.054
850	6		49	.058
900	6.5		63	.070
950	6.5		72	.076
975	6		84	.086
1000	6		100	.100
1025	6		108	.105
1050	4		133	.127
1075	4.5		160	.149
1100	0.5		187	.170
1125	-8		250	.222

Gage 1 shows a possible variation of the data. At moderate pressures some term of the series other than the first predominates. As the critical pressure is approached the principal mode of distortion predominates. The data are not plotted since no trend can be determined.

Data from Gage 2 are plotted in Fig. 8.

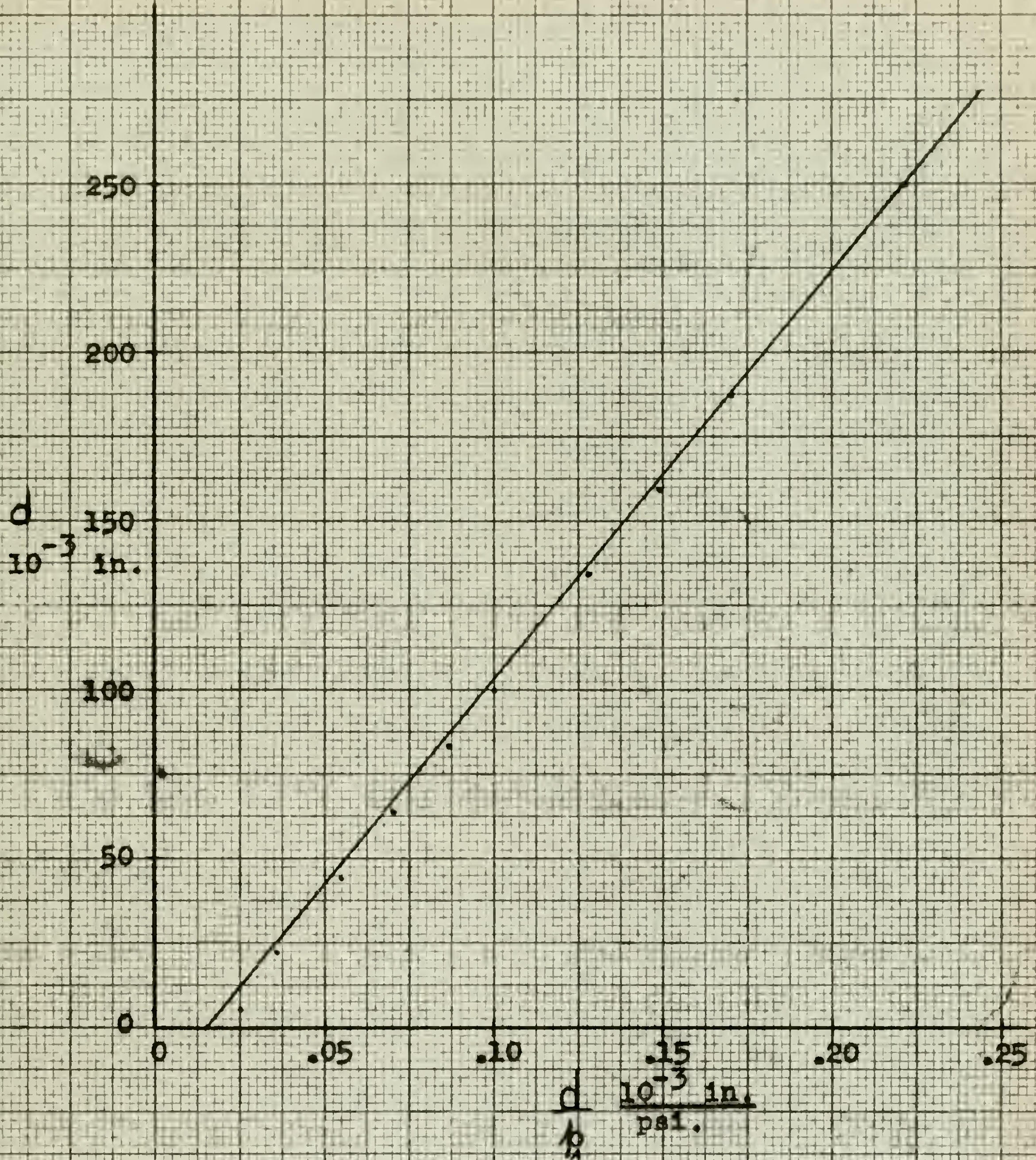

Figure 8. Data of Run 1, Gage 2.

Column 11 Run 2

	Gage 1		Gage 2	
p	d	d/p	d	d/p
200	1	.0050	3	.015
400	3.5	.0088	8	.020
600	7.5	.0125	18	.030
700	9.5	.0136	24	.034
800	15	.0187	34	.042
850	16.5	.0194	41	.048
900	18.5	.0210	49	.056
950	22.5	.0237	64	.067
975	23.5	.0241	69	.071
1000	26.5	.0260	83	.083
1025	29.5	.0288	95	.093
1050	32.5	.0310	110	.105
1075	36.5	.0340	129	.120
1100	39.5	.0360	155	.141
1125	41.5	.0369	173	.154
1150	47.5	.0413	250	.218

Data from Gage 1 are plotted in Fig. 9.

Data from Gage 2 are plotted in Fig. 10.

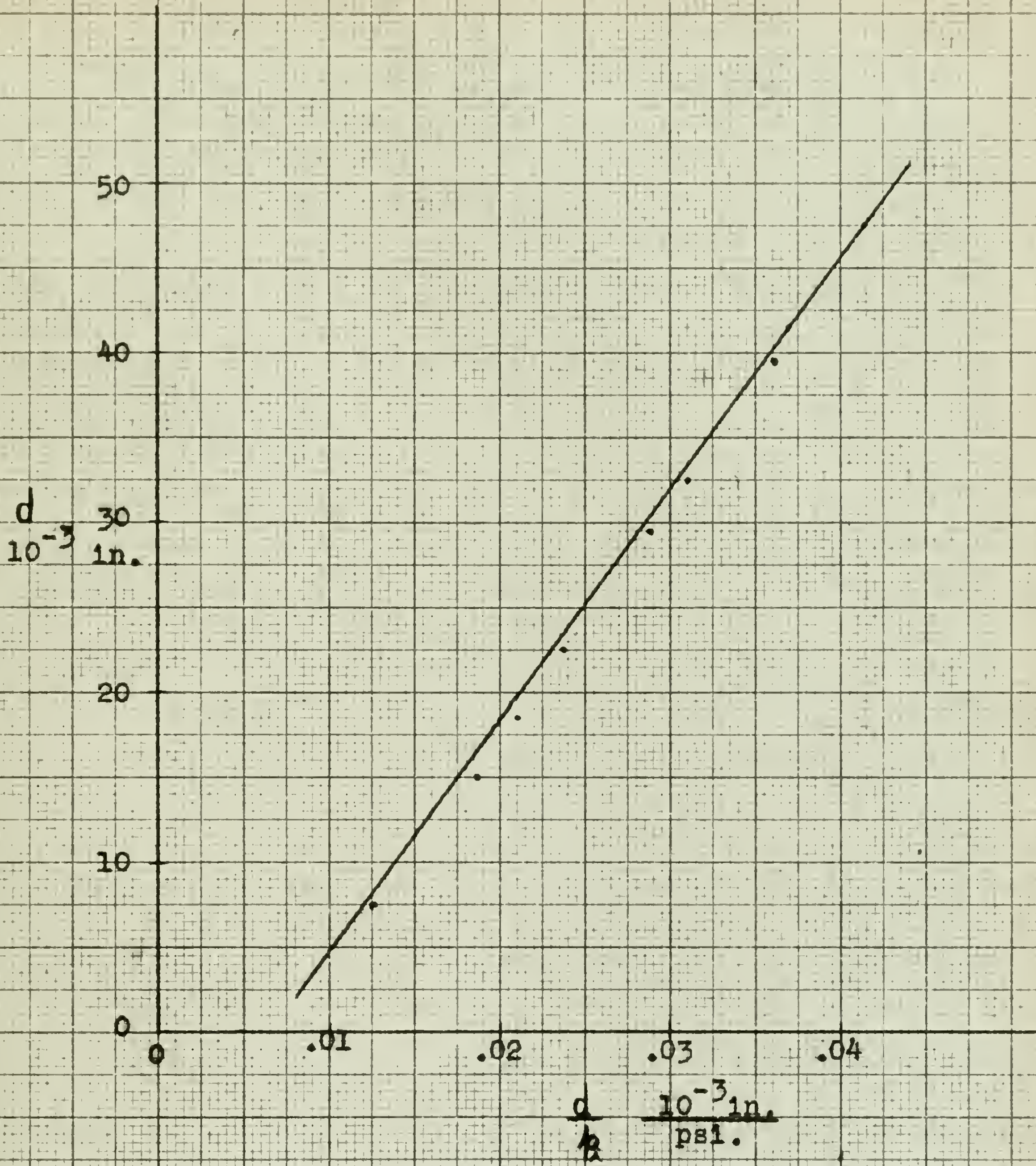

Figure 9. Data of Run 2, Gage 1.

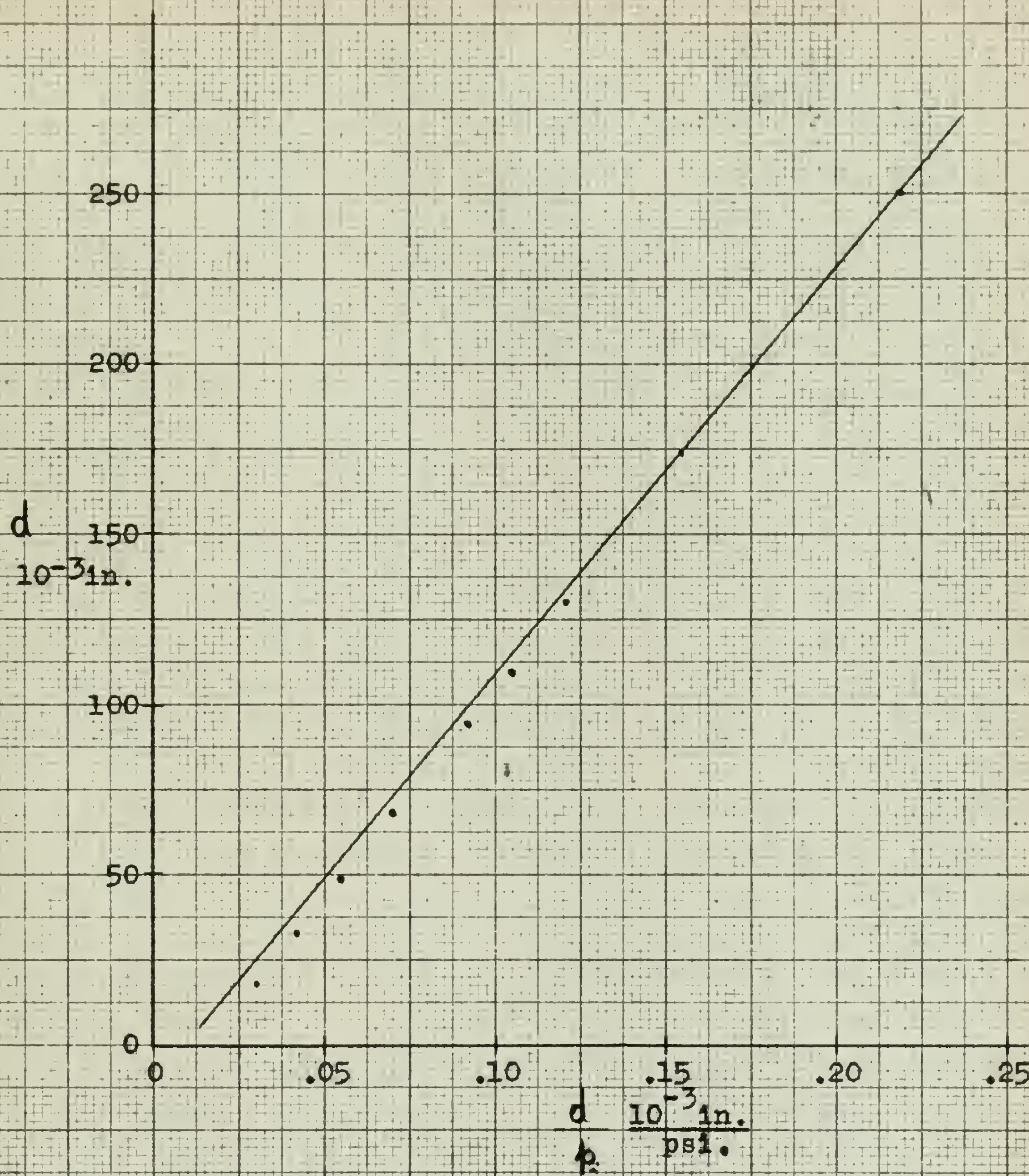

Figure 10. Data of Run 2, Gage 2.

Column 11 Run 3

	Gage 1		Gage 2	
$p_\wedge$	d	$d/p_\wedge$	d	$d/p_\wedge$
200	0		5	.025
400	1		12.5	.031
600	1		23.5	.039
700	0		32.5	.046
800	0		43.5	.054
850	0		53.5	.063
900	0		63.5	.071
950	-1	.0015	73.5	.078
975	-3	.0030	88.5	.091
1000	-3	.0030	93.5	.094
1025	-5	.0049	106.5	.104
1050	-8	.0076	138.5	.132
1075	-11	.0102	164.5	.153
1100	-15.5	.0141	171.5	.156
1125	-26	.0231	236.5	.210

Data from Gage 1 are plotted in Fig. 11.

Here is an example of the first term of the series predominating as the critical pressure is approached. In this case a trend is established.

Data from Gage 2 are plotted in Fig. 12.

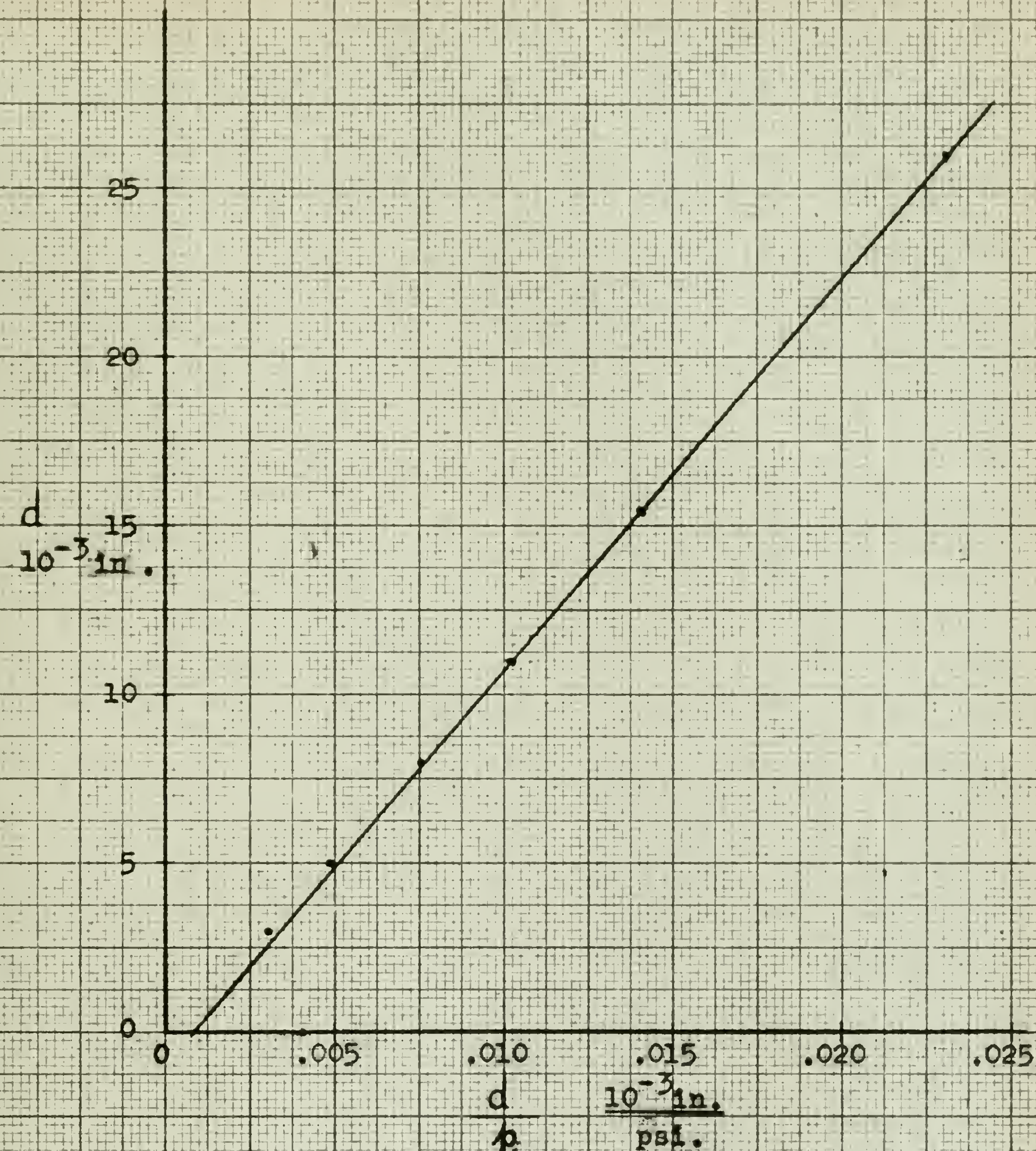

Figure 11. Data of Run 3, Gage 1.

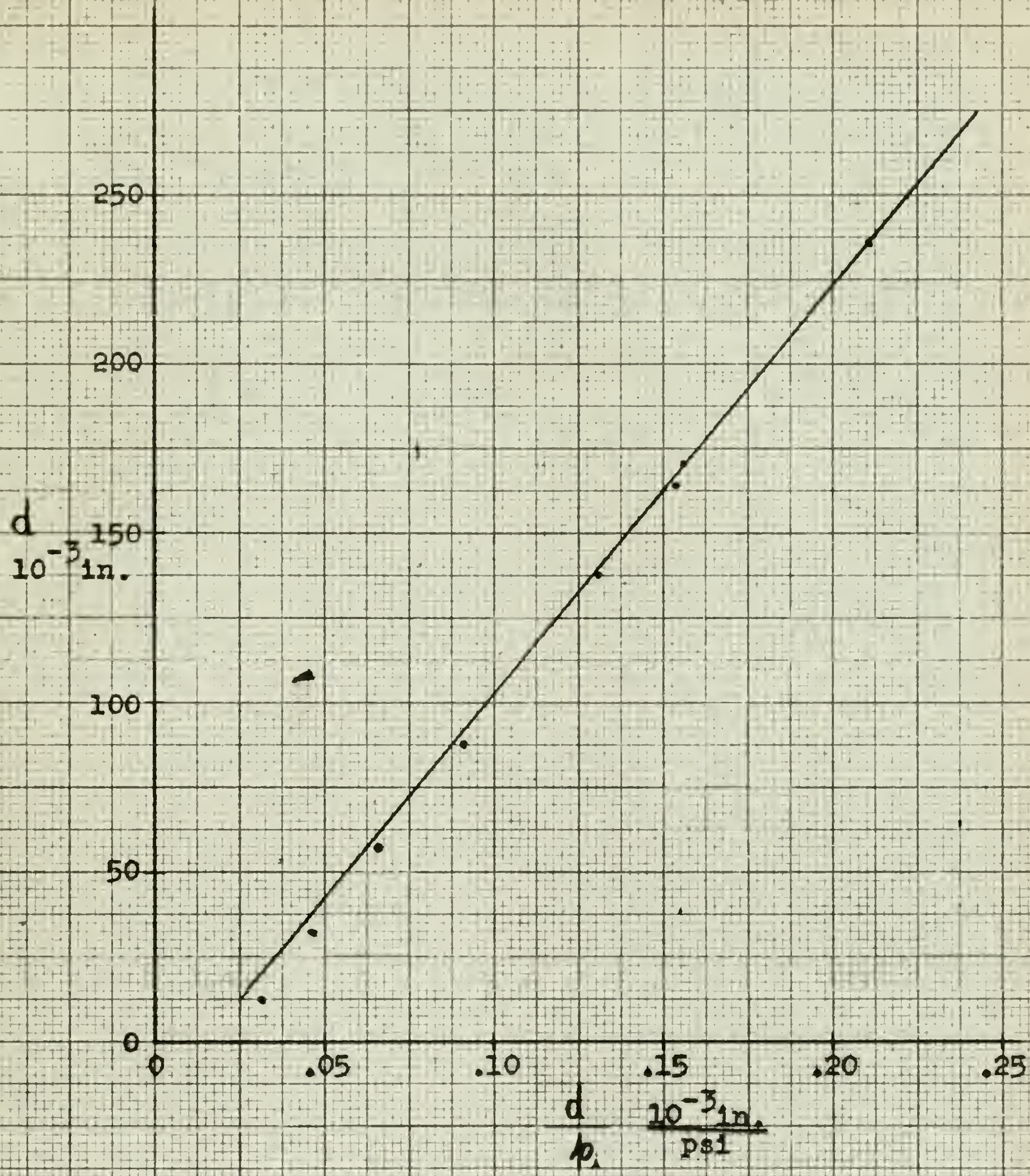

Figure 12. Data of Run 3, Gage 2.

3. Criteria Used in Evaluating Critical Pressures.

Let the points obtained from the data be numbered "1" through "n" in order of increasing load. For a "good" data set the following generalization can be made:

1) The slope of the line joining points "n-2" and "n" will be approximately equal to the slope of the line joining points "n - 1" and "n."

2) The best estimate of the asymptotic slope, without extrapolation, is the slope of the line joining points "n - 1" and "n."

Table 2 lists the slopes obtained from the last three points of Figs. 8 through 12

Table 2

Slopes Obtained from Data Sets of Column 11

Run 1	
Gage 1	Gage 2
No curve	1231
	1210
Run 2	
Gage 1	Gage 2
1510	1233
1362	1202
Run 3	
Gage 1	Gage 2
1163	1262
1165	1202

The following criteria were set up to evaluate the information obtained from the plots. Data sets in which the deflection changes direction within the last three readings were discarded without plotting. Data sets in which the two slopes computed from the last three points differed by more than five per cent were discarded. Data sets in which the two slopes agreed within five per cent were assigned a weight of unity. Data sets in which the two slopes were within one per cent were assigned a weight of 2. The weighted average of the slopes of the lines joining the last two points was taken as the critical pressure of the column.

The above system of evaluation, though it does not prevent discarding good information, makes it unlikely that poor information will be included.

4. Tabulation of geometrical properties and critical pressures.

The geometric properties, critical pressures, and average critical pressure for each column are given in Table 3. Method of determining theoretical values used in Table 3 may be found in the section of Appendix I indicated:

Geometric Properties	Section 5
Critical Pressures	Section 3
Average Critical Pressure	Section 3

Table 3. Tabulation of Geometric Properties, Critical Pressures and Average Critical Pressure for Runs Used

Column	Length (in.)	(in.)		Area (in.^{2})	Critical Pressure (psi)						Average Critical Pressure (psi)
1	26.60	.253	105.14	.079090	1076	1053	1093	1082	1066		1074
2	26.60	.253	105.14	.079119	1110	1127	1072*	1093	1075		1091
**3	30.53	.254	120.20	.079224	819	821					820
4	30.53	.254	120.20	.079229	820	790	822	843			819
5	37.53	.254	147.76	.079306	537	520	587	546	519		543
6	37.53	.254	147.83	.079209	543	543					543
7	51.65	.255	202.53	.079615	296	284	308				296
8	51.60	.255	202.35	.079685	285	303	287				292
9	34.46	.343	100.47	.108583	1637	1580	1672	1610			1625
10	34.46	.343	100.47	.108823	1642	1700	1612				1649
11	40.46	.341	118.65	.108032	1210	1202	1165	1202			1189
12	40.52	.343	118.13	.108765	1152	1201	1198	1220			1194
13	48.47	.341	142.17	.108031	816	793	804	800	782		800
14	48.52	.341	142.29	.108163	810	814	801	800	834		811
15	63.44	.342	185.58	.107018	464	466	458	472			465
16	63.45	.342	185.53	.107082	487	469	466	492	475	466	474

17	43.52	.429	101.45	.139415	2010	2010	2012	2021			2012
18	43.52	.429	101.45	.139464	2008	1955	2070	2090	1940		2024
19	Tube wall split when pressure applied										
20	50.63	.429	118.02	.137605	1475	1490	1530				1503
21	58.46	.429	146.27	.138441	1119	1128	1121	1119	1118	1120	1120
22	58.46	.429	146.27	.138141	1070	1160	1130				1118
23	65.51	.430	152.35	.139441	913	923	927	910			917
24	65.46	.430	152.35	.139418	918	916	925	906			917

*Weight factors greater than unity underlined.
**Permanently deformed on first run.

5. Measured Quantities

1. The length of the column was measured with a 6 foot steel tape. It is expected that measurements are within .03 inch.

2. The outer diameter and thickness were measured with a micrometer. It is expected that measurements are within .0005 inch.

3. The radius of gyration was obtained analytically using the outer diameter and thickness.

4. The weight of the column was obtained to the nearest 0.1 gram.

5. The area was obtained analytically by dividing the tube weight by the product of the length by the density. The tabulated density of 24S-T3 aluminum alloy is 0.100 pounds per cubic inch.

6. The tabulated value of the modulus of elasticity of 24S-T3 aluminum alloy is 10.6×10^6 lbs per $inch^2$. This value was verified experimentally.

7. The theoretical critical pressure was computed using a calculator. An excess number of figures was carried to avoid round-off errors.